Polynomial Selection

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Given $N \in \mathbb{Z}$

Find co-prime polynomials $f, g \in \mathbb{Z}[x]$ with common zero modulo $N$

Degrees and coefficients as small as possible
Brief summary of polynomial selection

Given $N \in \mathbb{Z}$

Find co-prime polynomials $f, g \in \mathbb{Z}[x]$ with common zero modulo $N$

Degrees and coefficients as small as possible

Restriction to $\deg(f) = d$, $\deg(g) = 1$

Easy: coefficients of size $N^{\frac{1}{d+1}}$:

Choose $m = [N^{\frac{1}{d+1}}] + 1$, set $g = x - m$, $f = \sum_{i=0}^{d} a_i x^i$ where

$N = \sum_{i=0}^{d} a_i m^i$ is the base-$m$-expansion of $N$. 
Skewness:

Change sieving area from $-A \leq a \leq A, 0 < b \leq A$ to $-A\sqrt{s} \leq a \leq A\sqrt{s}, 0 < b \leq \frac{A}{\sqrt{s}}$ for some $s$ (skewness)

$\Rightarrow$ want to minimise $\max(|a_i| \cdot s^{i-\frac{d}{2}})$ \quad (f = \sum_{i=0}^{d} a_i x^i)$
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\(-A \sqrt{s} \leq a \leq A \sqrt{s}, 0 < b \leq \frac{A}{\sqrt{s}}\) for some \(s\) (skewness)

\[\Rightarrow \text{want to minimise } \max(|a_i| \cdot s^{i - \frac{d}{2}}) \quad (f = \sum_{i=0}^{d} a_i x^i)\]

Choose \(a_d\) smaller than \(N^{\frac{1}{d+1}}\), choose \(m\) near \(\left(\frac{N}{a_d}\right)^{\frac{1}{d}}\)
\[\Rightarrow |a_{d-1}| \text{ roughly of size } a_d, \text{ small enough}\]

Remaining coefficients of size \(\left(\frac{N}{a_d}\right)^{\frac{1}{d}}\)
ok for \(a_0, a_1\) (perhaps also for \(a_2\))

Coefficients \(a_{d-2}, \ldots, a_3, (a_2)\) too big \(\quad\) biggest problem \(a_{d-2}\)
Motivation

Lattice sieving for 768 bit numbers:
e.g.: factor base bounds $1.1 \cdot 10^9$ (for $f$), $2 \cdot 10^8$ (for $g$)
⇒ ca. 67 million factor base elements

gnfs-lasieveI16e needs 20 byte per factor base element:
• prime ideal $(p, x - r)$: 4 byte for $p$ and 4 byte for $r$
• two vectors in special $q$ lattice: $2 \cdot 4$ byte
• current location in special $q$ lattice: 4 byte

could reduce this:
• use 1 byte for storing differences of $p$ ⇒ 17 byte
• handle larger $p$ in a different way ⇒ 15 or 16 byte

How can we reduce this further?
If skewness were equal to size of sieving area:

form of sieving area: \(-A \leq a \leq A, b = 1\) (one line)
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Storage requirements for lattice siever (12 byte per factor base element):
- prime ideal \((p, x - r)\): 4 byte for \(p\) and 4 byte for \(r\)
- current location in special \(q\) lattice: 4 byte

We can
- recalculate \(r\) from last location in special \(q\) lattice \(\Rightarrow\) 8 byte
- store 1 byte differences of primes \(\Rightarrow\) 5 byte

Reduced storage for factor base from 1GB (or 1.3GB) to 350MB
How can we find such polynomials?
Polynomials with large skewness

Example: 768-bit integer $N$, size of sieving area $\approx 2^{64} \approx$ skewness,

$$f = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0, \quad g = lx - m$$

$$N = a_4 m^4 + a_3 lm^3 + a_2 l^2 m^2 + a_1 l^3 m + a_0 l^4$$
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<table>
<thead>
<tr>
<th>coefficient</th>
<th>$a_4$</th>
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<th>$a_2$</th>
<th>$a_1$</th>
<th>$a_0$</th>
<th>$l$</th>
<th>$m$</th>
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<tbody>
<tr>
<td>bit size</td>
<td>0</td>
<td>64</td>
<td>128</td>
<td>192</td>
<td>256</td>
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⇒ values of polynomials: ca. 256 bit and 192 bit
seems to be slightly worse than current degree 6 polynomials
**Polynomials with large skewness**

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$$f = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \ g = lx - m$$

$$N = a_4m^4 + a_3lm^3 + a_2l^2m^2 + a_1l^3m + a_0l^4$$

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⇒ values of polynomials: ca. 256 bit and 192 bit

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Check: $64 + 128 + 192 + 256 + 128 + 192 - 64 - 64 = 768 + 64$

⇒ expect to find $2^{64}$ such polynomial pairs

How can we find such polynomial pairs (with cost $\ll 2^{64}$)?
\[
\begin{align*}
f &= x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \quad g = lx - m \\
N &= m^4 + a_3lm^3 + a_2l^2m^2 + a_1l^3m + a_0l^4
\end{align*}
\]
\[ f = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \quad g = lx - m \]

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translation \(\Rightarrow\) can assume \(a_3 \in \{0, 1, 2, 3\}\)
\[ f = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0, \quad g = lx - m \]

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translation \( \Rightarrow \) can assume \( a_3 \in \{0, 1, 2, 3\} \)

Restrict to \( a_3 = 0 \), assume \( l \ll \frac{m}{2^{64}} \):

\[ f = x^4 + a_2 x^2 + a_1 x + a_0, \quad g = lx - m: \]

\[ N = m^4 + a_2 l^2 m^2 + a_1 l^3 m + a_0 l^4 = m^4 + l^2 R \quad \quad a_2 \approx \frac{R}{m^2} \]

New problem: to find \( l, m \) such that \( l^2 |N - m^4\) and \( \frac{|N-m^4|}{l^2 m^2} \) is small
General problem: $N$, $d$ and bound $B$ given, find $l$, $m$ such that $l^2|N - m^d$ and $\frac{|N - m^d|}{l^2 m^{d-2}} < B$
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Set $m_0 = \sqrt[d]{N}$, $m = m_0 + i$, $i \in [-M, M]$
$\Rightarrow |N - m^d| \lesssim d M m_0^{d-1}$

want $i$, $l$ such that $l^2|N - (m_0 + i)^d|$ and $\frac{d M m_0}{l^2} < B$
General problem: $N$, $d$ and bound $B$ given, find $l$, $m$ such that
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Set $l = p_1p_2, \; p_i \in \mathcal{P} \text{ primes, } \mathcal{P} = [P, 2P]$

1. generate pairs $(p, i)$ such that $p^2 |N - (m_0 + i)^d$
2. sort pairs w. r. t. second entry
3. for each collision, i. e., pairs $(p_1, i)$, $(p_2, i)$ with $p_1 \neq p_2$: output $l = p_1p_2$, $m = m_0 + i$

result: $|a_{d-2}| \approx \frac{|N - m^d|}{l^2 m^{d-2}} \lesssim \frac{dM}{P^4} m_0$
Analysis

\[ m_0 = \sqrt[\prime]{N}, \quad m = m_0 + i, \quad i \in [-M, M] \]

\[ l = p_1p_2, \quad p_i \in \mathcal{P} \text{ primes, } \mathcal{P} = [P, 2P] \]

number of pairs \( \approx \frac{M}{P \log P} \), number of collisions \( \approx \frac{M}{4P^2 (\log P)^2} \)
Analysis

\[ m_0 = \sqrt[4]{N}, \ m = m_0 + i, \ i \in [-M, M] \]

\[ l = p_1 p_2, \ p_i \in \mathcal{P} \text{ primes, } \mathcal{P} = [P, 2P] \]

number of pairs \( \approx \frac{M}{P \log P} \), number of collisions \( \approx \frac{M}{4P^2 (\log P)^2} \)

cost \( O\left(\frac{M \log M}{P \log P} + \frac{P}{\log P}\right) \)

result: \( |a_{d-2}| \lesssim \frac{dM}{P^4} m_0 \)
Analysis

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cost \( O\left(\frac{M \log M}{P \log P} + \frac{P}{\log P}\right) \)

result: \( |a_d - 2| \approx \frac{dM}{P^4} m_0 \)

for 768 bit example choose \( M = 2^{90}, P = 2^{39} \):
\[ \approx 1 \text{ collision, } \frac{dM}{P^4} m_0 \approx 2^{128}, \text{ cost } 2^{46} \text{ pairs} \]
Analysis

\( m_0 = \sqrt[4]{N}, \ m = m_0 + i, \ i \in [-M, M] \)

\( l = p_1p_2, \ p_i \in \mathcal{P} \) primes, \( \mathcal{P} = [P, 2P] \)

number of pairs \( \approx \frac{M}{P \log P} \), number of collisions \( \approx \frac{M}{4P^2(\log P)^2} \)

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result: \( |a_{d-2}| \approx \frac{dM}{P^4} m_0 \)

for 768 bit example choose \( M = 2^{90}, P = 2^{39} \):

\( \approx 1 \) collision, \( \frac{dM}{P^4} m_0 \approx 2^{128} \), cost \( 2^{46} \) pairs

choosing \( M = P^2 \):

cost per collision \( O(P(\log P)^2) \), result \( |a_{d-2}| \approx \frac{d}{P^2} m_0 \)
Asymptotic considerations

degree \( d = \left( \frac{3 \log N}{\log \log N} \right)^{\frac{1}{3}} \), sieving area \( \approx L\left(\frac{1}{3}, \sqrt[3]{\frac{64}{9}}\right) \approx \text{skewness} \)

product of coefficient ranges of algebraic polynomial = \( L(1, \frac{7}{8}) \)
⇒ cannot find such polynomial pairs

Remark: polynomial pairs of degree \( d \) and \( d - 1 \) would be ok
General situation

\[ N = a_d m^d + a_{d-1} l m^{d-1} + l^2 R \]

Find \( l, m \) such that and \( \frac{|R|}{m^{d-2}} \approx |a_{d-2}| \) is sufficiently small.
General situation

\[ N = a_d m^d + a_{d-1} l m^{d-1} + l^2 R \]

Find \( l, m \) such that and \( \frac{|R|}{m^{d-2}} \) (≈ \( |a_{d-2}| \)) is sufficiently small.

Reduction to \( a_d = 1, a_{d-1} = 0 \) (translation \( x \mapsto x - \frac{a_{d-1}}{d a_d} \)):

\[ d^d a_d^{d-1} N = (d a_d m + a_{d-1} l)^d + l^2 \left( d^d a_d^{d-1} R - (d a_d m)^{d-2} \cdot \binom{d}{2} \cdot a_{d-1}^2 - \ldots \right) \]
General situation

\[ N = a_d m^d + a_{d-1} l m^{d-1} + l^2 R \]

Find \( l, m \) such that \( \frac{|R|}{m^{d-2}} \) \((\approx |a_{d-2}|)\) is sufficiently small.

Reduction to \( a_d = 1, a_{d-1} = 0 \) (translation \( x \mapsto x - \frac{a_{d-1}}{d a_d} \)):

\[ d^d a_{d-1}^d N = (d a_d m + a_{d-1} l)^d + l^2 \left( d^d a_{d-1}^d R - (d a_d m)^{d-2} \cdot \binom{d}{2} \cdot a_{d-1}^2 - \ldots \right) \]

or

\[ \tilde{N} = \tilde{m}^d + l^2 \tilde{R} \quad \text{where} \quad \tilde{N} = d^d a_{d-1}^d N, \tilde{m} = d a_d m + a_{d-1} l \]
General situation

\[ N = a_d m^d + a_{d-1} l m^{d-1} + l^2 R \]

Find \( l, m \) such that and \( \frac{|R|}{m^{d-2}} \approx |a_{d-2}| \) is sufficiently small.

Reduction to \( a_d = 1, \ a_{d-1} = 0 \) (translation \( x \mapsto x - \frac{a_{d-1}}{da_d} \)):

\[ d^d a_d^{d-1} N = (d a_d m + a_{d-1} l)^d + l^2 \left( d^d a_d^{d-1} R - (d a_d m)^{d-2} \cdot \binom{d}{2} \cdot a_{d-1}^2 - \ldots \right) \]

or

\[ \tilde{N} = \tilde{m}^d + l^2 \tilde{R} \quad \text{where} \quad \tilde{N} = d^d a_d^{d-1} N, \tilde{m} = d a_d m + a_{d-1} l \]

1. find \( l, \tilde{m} \) as above

2. \( \tilde{m} = d a_d m + a_{d-1} l \): find \( m, \ 0 \leq a_{d-1} < da_d \quad (\gcd(l, da_d) = 1) \)

Result:

\[ |a_{d-2}| \approx \frac{|\tilde{R}|}{d^2 a_d \tilde{m}^{d-2}} \lesssim \frac{d M \tilde{m}_0}{d^2 a_d P^4} \approx \frac{M}{P^4} m_0 \]
Some tricks

Replace \( l = p_1p_2 \) by \( l = cp, \ c \in \mathcal{C}, \ p \in \mathcal{P} \)

E. g.: \( \mathcal{C} = [P_1, P_2], \ \mathcal{P} = \{ p \in [P_2, P_3]| \text{p prime}\} \) for some \( P_1 < P_2 < P_3 \)

1. generate pairs \((c, i), \ c \in \mathcal{C}\)
2. generate pairs \((p, j), \ p \in \mathcal{P}\)
3. search for collisions between \(c\)-pairs and \(p\)-pairs, and for collisions within \(p\)-pairs

many alternative approaches, e. g.:

- arbitrary \( \mathcal{C}, \mathcal{P} \), remove multiples of primes of \( \mathcal{P} \) from \( \mathcal{C} \)

- \( \mathcal{C} = \{ c \in [P_1, P_2]|p|c \Rightarrow p \equiv 1 \ (\text{mod} \ 4)\} \),
  \( \mathcal{P} = \{ c \in [P_1, P_2]|p|c \Rightarrow p \equiv 3 \ (\text{mod} \ 4)\} \)

- ...
Special $q$

Choose $q$, $0 \leq s < q^2$ such that $q^2 | N - (m_0 + s)^d$

Search for $l'$ with $l'^2 | N - (m_0 + s + iq^2)^d$ as above and set $l = l'q$

analysis remains the same, only $l$ is increased by $q$

Advantage: Initialisation costs drop, since expensive root calculation of $N - x^d$ modulo $p$ (resp. $c$) can be used for many $q$

Even better: can do inversion modulo $p^2$ for many $q$ simultaneously

$\Rightarrow$ cost drops to a few modular additions + multiplications per generated pair
Some results

<table>
<thead>
<tr>
<th>number</th>
<th>sieving time</th>
<th>pol. sel. time</th>
<th>improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>RSA512</td>
<td>≈ 0.25 a</td>
<td>4 d, 4 d, 4 d</td>
<td>0.84, 0.8, 0.84</td>
</tr>
<tr>
<td>RSA576</td>
<td>≈ 2.5 a</td>
<td>15 d</td>
<td>0.87</td>
</tr>
<tr>
<td>RSA640</td>
<td>≈ 20 a</td>
<td>10 d</td>
<td>0.77 (?)</td>
</tr>
</tbody>
</table>

improvement = time for new pol. pair / time for old pol. pair

RSA512: comparison with best polynomial pair found by old method

RSA576, RSA640: comparison with polynomial pairs used in factorisation